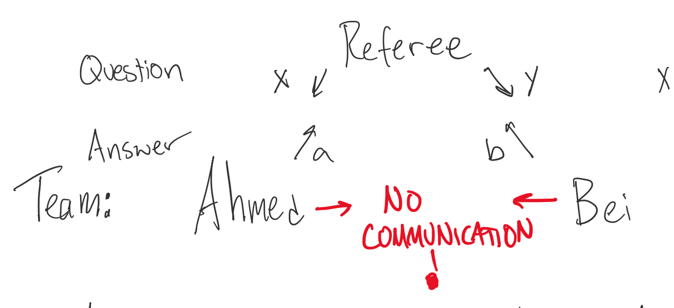
3: CHSH game

Monday, March 8, 2021 9:59 PM

Problem: Win a Collaborative Game!



$x, y, a, b \in \{0, 1\}$

BUT can agree on strategy ahead of time.

A + B win $\longleftrightarrow$ $a \oplus b = x \wedge y$

p	q	$p \oplus q$	$p \wedge q$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Q: What is Ahmed + Bei's best strategy if referee chooses x, y randomly?

A: A, B always set $a=b=0$, regardless of x, y. $\Rightarrow$ win 75% of time.

... Give each a qubit*! Do better?

... How to describe 2 qubits?

* Diamond nitrogen vacancy qubit $|0\rangle, |1\rangle$: magnetic spin of electron (up or down)

SUPERPOSITION, CHSH

Qubit A $\quad |\psi_1\rangle_A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

Qubit B $\quad |\psi_2\rangle_B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$

State of A + B

$$|\psi\rangle_{AB} = |\psi_1\rangle_A \otimes |\psi_2\rangle_B = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}$$

With kets / standard basis:

$\otimes$ distributes like regular multiplication

$$|+\rangle \otimes |1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) = \tfrac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

Notation

Implied tensor product: $|+\rangle|1\rangle = |+\rangle \otimes |1\rangle$

Standard basis Notation: $|0\rangle|1\rangle = |01\rangle$
ONLY

Why a 4×1 vector represents 2 qubits:

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

count in binary: 00, 01, 10 ←, 11

Q: What is $(|10\rangle)^\dagger$?

A) $\langle 10|$ ⇑ B) $\langle 01|$

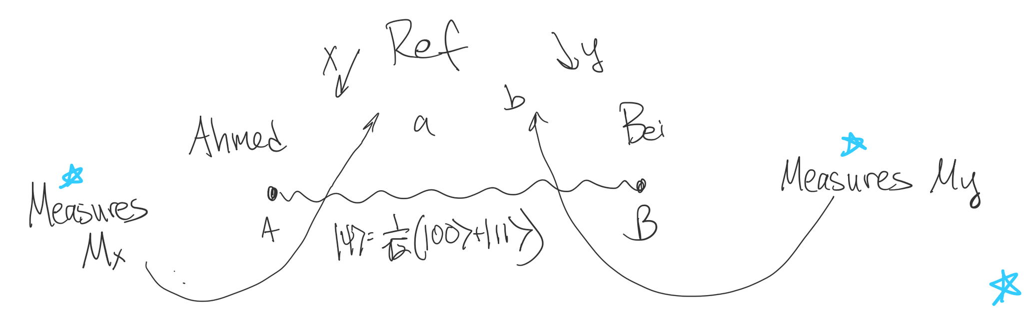
$$\langle 10| = \langle 1| \otimes \langle 0| = (0\ 1) \otimes (1\ 0) = (0\ 0\ 1\ 0)$$

$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

2 Qubit State

$$|\psi\rangle_{AB} = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} \qquad \sum_{i \in \{0,1\}^2} |a_i|^2 = 1$$

← A and B don't each have their own state



* Shine laser at qubit. Shines back if in $|0\rangle$, not if in $|1\rangle$

M_x, M_y are single qubit measurements

$M = M_x \otimes M_y$ combined measurement

ex: $M_x = \{|0\rangle, |1\rangle\}$ $\quad M_y = \{|+\rangle, |-\rangle\}$

$M_{AB} = \{|0\rangle_A|+\rangle_B, |0\rangle_A|-\rangle_B, |1\rangle_A|+\rangle_B, |1\rangle_A|-\rangle_B\}$

Orthonormal Basis! 4 outcomes

Measure $|\psi\rangle_{AB}$ with M_{AB}. If $|\phi\rangle$ is in M_{AB}, get

- Outcome $|\phi\rangle$ w/ prob $|\langle\phi|\psi\rangle|^2$. $|\psi\rangle \rightarrow |\phi\rangle$

ex: If measure $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ith M_{AB}, what is the probability of getting outcome $|0\rangle_A|+\rangle_B$? (A outcome is $|0\rangle$, B outcome is $|+\rangle$)

what is conj transpose

$$\left| \left(|0\rangle_A|+\rangle_B\right)^\dagger \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right|^2$$

$$= \tfrac{1}{2} \left| \langle 0|_A \langle +|_B \left(|00\rangle_{AB} + |11\rangle_{AB}\right) \right|^2$$

ORDER SAME!

$$= \tfrac{1}{2} \left| \langle 0|_A \langle +|_B |0\rangle_A |0\rangle_B + \langle 0|_A \langle +|_B |1\rangle_A |1\rangle_B \right|^2$$

$$= \tfrac{1}{2} \left| \langle 0|0\rangle_A \langle +|0\rangle_B + \langle 0|1\rangle_A \langle +|1\rangle_B \right|^2$$

($\langle 0|0\rangle = 1$, $\langle +|0\rangle = \frac{1}{\sqrt{2}}$, $\langle 0|1\rangle = 0$)

$$= \tfrac{1}{4}$$

Ahmed + Bei's Strategy:

Share state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$

Let $M(\omega) = \{|R(\omega)\rangle, |Q(\omega)\rangle\}$ where $|R(\omega)\rangle = \cos(\omega)|0\rangle + \sin(\omega)|1\rangle$
$|Q(\omega)\rangle = -\sin(\omega)|0\rangle + \cos(\omega)|1\rangle$

Ahmed

x	Measurement
0	$M(0)$
1	$M(\pi/4)$

Bei

y	Measurement
0	$M(\pi/8)$
1	$M(-\pi/8)$

Outcome	Response to Ref
R	0
Q	1

Measurement Outcome

Group	x	y	Ahmed	Bei
Even	0	0	R	R
ODD	0	0	Q	Q

} Good outcomes
⋮
What is probability of these outcomes

x	y	A	B
0	1	R	Q

$$\rightarrow |R(0)\rangle_A |Q(-\pi/8)_B\rangle \Rightarrow \left| \langle R(0)|_A \langle Q(\tfrac{-\pi}{8})|_B \tfrac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) \right|^2 \ldots$$

$$\rightarrow |R(0)\rangle_A |R(\pi/8)_B\rangle \Rightarrow \left| \langle R(0)|_A \langle R(\pi/8)|_B \tfrac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) \right|^2 \ldots$$

$$= \left| \langle 0| \left(\cos\tfrac{\pi}{8}\langle 0| + \sin\tfrac{\pi}{8}\langle 1|\right) \tfrac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) \right|^2$$

$$= \left| \left(\cos\tfrac{\pi}{8}\langle 00| + \sin\tfrac{\pi}{8}\langle 01|\right) \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \right|^2$$

✓ ✗ ✗ ✗

$$= \tfrac{1}{2}\cos^2\tfrac{\pi}{8}$$

85% > 75%! Can win the game more often!

Why?
- Verify quantumness (Bell Test)
- Quantum randomness